Torsional oscillations of an infinite plate in second-order fluids

By A. C. SRIVASTAVA
Department of Mechanical Engineering, University of Illinois, Urbana, Illinois, U.S.A.†

(Received 4 March 1963)

The flow of an incompressible second-order fluid due to torsional oscillations of an infinite plate when the fluid is infinite in extent as well as the case when it is bounded by another stationary parallel plate has been considered by expanding the velocity components and the pressure in powers of the amplitude of oscillation of the plate. In both cases the first-order solution consists of a transverse velocity and the second-order solution gives a radial-axial flow composed of a steady part and a fluctuating part. In the case of the unbounded plate the steady part of the radial flow does not vanish outside the boundary-layer region. Hence the equations are solved by another approximate method for the steady part of the flow. The effects of the non-Newtonian terms are to increase the non-dimensional boundary thickness and the shearing stress on the plate. In the case of two plates the velocity components and the shearing stresses on the plates have been expressed in powers of Reynolds number R for its small values. Their asymtotic behaviour for large R has also been studied. The asymtotic expansion of the fluctuating part of the radial-axial flow shows that the boundary layer is developed at both the plates.

1. Introduction

A *simple material* is a substance for which the stress is determined by a knowledge of the entire history of the strain. A simple material is called a *simple fluid* if it has the property that all local states with the same mass density are intrinsically equivalent in response, with all observable differences in response being due to definite differences in history (Noll 1958). For any given history $g(s)$ a *retarded history* $g_\alpha(s)$ can be defined as

$$g_\alpha(s) = g(\alpha s) \quad (0 \leqslant s < \infty),$$

where α is *the retardation factor,* $0 < \alpha \leqslant 1$. Taking this definition of retarded history and assuming that the stress is more sensitive to recent deformation than to deformations which occurred in the distant past, Coleman & Noll (1960) proved that the theory of simple fluids yields the theory of perfect fluids (deviatoric stress is independent of strain-rate) for $\alpha \to 0$ and yields the theory of Newtonian fluids (deviatoric stress is linearly proportional to deviatoric strain-rate) as the next

† Permanent address: Department of Mathematics, Indian Institute of Technology, Kharagpur, India.

approximation. The theory of Newtonian fluids gives a correction to the theory of perfect fluids which is complete to within terms of order one in α. If we neglect all the terms of order greater than two in α the constitutive equation of an incompressible simple fluid can be written as

$$\tau_{ij}+p\delta_{ij} = \mu_1 A_{(1)ij}+\mu_2 A_{(2)ij}+\mu_3 A_{(1)ik} A_{(1)kj}, \quad (1)$$

where $$A_{(1)ij} = v_{i,j}+v_{j,i}, \quad A_{(2)ij} = a_{i,j}+a_{j,i}+2v_{m,i}v_{m,j}, \quad (2)$$

τ_{ij} is the stress tensor, v_i, a_i are the velocity and the acceleration vectors, μ_1, μ_2, μ_3 are the material constants and p is an indeterminate hydrostatic pressure. The fluid governed by this constitutive equation is called an *incompressible second-order fluid*, for it gives complete second-order corrections to the incompressible perfect fluid. The constitutive equation of the general Rivlin–Ericksen fluids (1955) also reduces to (1) when square and products of $A_{(2)ij}$ are neglected and the coefficients of the remaining terms are taken to be constants. The solution of poly-iso-butylene in cetane behaves as a second-order fluid and the values of the constants μ_1, μ_2 and μ_3 have been determined by H. Markovitz (unpublished).

The torsional oscillation of a plate in Newtonian fluids has been discussed by Rosenblat (1959). He obtained the solution by expanding velocity components and the pressure in powers of the amplitude of the oscillation of the plate and showed that the solution is highly convergent within the boundary layer. He has also discussed the case when the fluid is confined between two torsionally oscillating plates (Rosenblat 1960). Similar problems in Reiner–Rivlin fluids were discussed by the author (Srivastava 1959, 1960). The aim of the present paper is to study these problems in second-order fluids. Assuming the amplitude of the oscillation of the plate to be small, the flow parameters and the pressure are expanded in its powers. The non-Newtonian effects are exhibited through two dimensionless parameters $\alpha\,(=\mu_2 n/\mu_1)$ and $\beta\,(=\mu_3 n/\mu_1)$, n being the frequency of the oscillation of the plate. By putting $\alpha = \beta = 0$ in any expression the corresponding expression for the flow of Newtonian fluid is reproduced. In both cases, that is when the fluid is infinite in extent and when it is bounded by another stationary parallel plate, the first-order solution consists of a transverse velocity which is independent of β. The second-order solution gives a radial-axial flow which involves both α and β and is composed of a steady term and a term of frequency $2n$. In the case of oscillation of an unbounded plate the steady part of the radial flow persists outside the boundary-layer region. This unexpected behaviour is the consequence of the approximation of the inertia force by its centrifugal part only in the series expansion. This is in no way justified outside the boundary-layer region where this part is vanishingly small and the neglected part of the inertia force is at least comparable to it. Hence the complete equations are solved by another approximate Pohlhausen-type method. The graphs of the steady part of the radial flow and the axial flow for a 6·8% solution of poly-iso-butylene in cetane at 30 °C (for n = 1·0, 1·5, 2·0) have been drawn (figures 1 and 2) and just for comparison the lines representing the corresponding flow neglecting the non-Newtonian effects are also drawn in the graphs. In the case when the fluid is confined between an oscillating and a stationary plate the velocity components and the shearing stresses on the plates have been expressed in powers of the Reynolds number R ($= nd^2/\nu_1$) for its small values and their asymtotic behaviour for large R has also been found. The transverse velocity behaves as if the stationary plate were absent when R tends to infinity. The transverse shearing stress on the stationary plate tends to that on the oscillating plate when R tends to zero, but when R tends to infinity it tends to zero. The unsteady part of the radial-axial flow suggests that the boundary layer is developed at both the plates for large values of R.

2. The equations of motion

The equations of motion in cylindrical polar co-ordinates (r, θ, z) with azimuthal variations neglected are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right) = \frac{\partial \tau_{rr}}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r}, \tag{3}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right) = \frac{\partial \tau_{\theta r}}{\partial r}+\frac{\partial \tau_{\theta z}}{\partial z}+2\frac{\tau_{\theta r}}{r}, \tag{4}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) = \frac{\partial \tau_{zr}}{\partial r}+\frac{\partial \tau_{zz}}{\partial z}+\frac{\tau_{zr}}{r}, \tag{5}$$

where ρ is the density and u, v, w are the velocity components in the directions of r, θ, z respectively. The equation of continuity is

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0. \tag{6}$$

Consider an infinite plate ($z = 0$) performing rotatory oscillations of the type $r\Omega\cos(nt)$ and of small amplitude ϵ about an axis ($r = 0$) perpendicular to its plane in an incompressible second-order fluid. When the fluid fills all the space $z > 0$ the boundary conditions can be written as

$$\left.\begin{aligned} &u = 0, \quad v = r\Omega\cos(nt), \quad w = 0 \quad \text{at} \quad z = 0,\\ &u \to 0, \quad v \to 0 \qquad\qquad\qquad\quad \text{as} \quad z \to \infty.\end{aligned}\right\} \tag{7}$$

In presence of another stationary parallel plate at $z = d$ the boundary conditions become

$$\left.\begin{aligned} &u = 0, \quad v = r\Omega\cos(nt), \quad w = 0 \quad \text{at} \quad z = 0,\\ &u = 0, \quad v = 0, \qquad\qquad\quad w = 0 \quad \text{at} \quad z = d.\end{aligned}\right\} \tag{8}$$

In both the cases we assume the velocity components and the pressure to be of the following form:

$$u = r\Omega F'(\eta, T), \quad v = r\Omega G(\eta, T), \quad w = -2d\Omega F(\eta, T), \tag{9}$$

$$p = \mu_1\Omega[-p_1+(r^2/d^2)p_2], \tag{10}$$

where $$Z = d\eta, \quad t = T/n, \quad \nu_1 = \mu_1/\rho, \quad \Omega = n\epsilon,$$

and a prime denotes differentiation with respect to η. In the case of a single unbounded plate the distance d is defined as $(2\nu_1/n)^{\frac{1}{2}}$. Substituting these expressions for the velocity components and the pressure in the equations (1)–(5) and taking $R = nd^2/\nu_1$, $\alpha = \mu_2 n/\mu_1$, $\beta = \mu_3 n/\mu_1$, we get

$$R[\partial F'/\partial T + \epsilon(F'^2 - G^2 - 2FF'')] = F''' + \alpha[\partial F'''/\partial T - 2\epsilon(FF^{iv} - F''^2)] + \beta\epsilon(F''^2 - G'^2 - 2F'F''') - 2p_2, \quad (11)$$

$$R[\partial G/\partial T + 2\epsilon(F'G - FG')] = G'' + \alpha[\partial G''/\partial T + 2\epsilon(F''G' - FG''')] + 2\beta\epsilon(F''G' - F'G''), \quad (12)$$

$$R[-2\,\partial F/\partial T + 4FF'] = p_1' - 2F'' + 2\alpha[-\partial F''/\partial T + 2\epsilon(9FF'' + FF''')] + 28\beta\epsilon F'F'' + (r^2/d^2)[(4\alpha + 2\beta)(F''F''' + G'G'') - p_2']. \quad (13)$$

Equating the terms independent of r and the coefficient of r^2 on both sides of (13) we get two equations. Integrating the equation arising out of the terms independent of r we get p_1, while integrating the second one we have

$$p_2 = (2\alpha + \beta)(F''^2 + G'^2) + \lambda. \quad (14)$$

In the case of a single plate when fluid fills the space $z > 0$ the constant of integration λ is zero in view of the condition

$$F'' \to 0, \quad G' \to 0, \quad p_2 \to 0 \quad \text{as} \quad \eta \to \infty. \quad (15)$$

With this value of p_2 (11) becomes

$$R[\partial F'/\partial T + \epsilon(F'^2 - G^2 - 2FF'')] = F''' + \alpha[\partial F'''/\partial T - 2\epsilon(F''^2 + 2G'^2 + FF^{iv})] - \beta\epsilon(F''^2 + 3G'^2 + 2F'F''') - 2\lambda. \quad (16)$$

The functions F, G and the constant of integration λ can be completely determined from (12), (16) and the boundary conditions. A solution is sought here by expanding them in powers of ϵ. Substituting the series

$$F(\eta, T) = F_0(\eta, T) + \epsilon F_1(\eta, T) + \epsilon^2 F_2(\eta, T) + \ldots,$$
$$G(\eta, T) = G_0(\eta, T) + \epsilon G_1(\eta, T) + \epsilon^2 G_2(\eta, T) + \ldots,$$
$$\lambda(T) = \lambda_0(T) + \epsilon\lambda_1(T) + \epsilon^2\lambda^2(T) + \ldots,$$

into (16) and (12) and equating the coefficients of like powers of ϵ we obtain the following system of linear partial differential equations:

$$R\,\partial F_0'/\partial T = F_0''' + \alpha\,\partial F_0'''/\partial T - 2\lambda_0, \quad (17)$$

$$R[\partial F_1'/\partial T + (F_0'^2 - G_0^2 - 2F_0F_0'')] = F_1''' + \alpha[\partial F_1'''/\partial T - (2F_0''^2 + 4G_0'^2 + 2F_0F_0^{iv})] - \beta(F_0''^2 + 3G_0'^2 + 2F_0'F_0''') - 2\lambda_1, \quad (18)$$

$$R\,\partial G_0/\partial T = G_0'' + \alpha\,\partial G_0''/\partial T, \quad (19)$$

$$R[\partial G_1/\partial T + 2(F_0'G_0 - F_0G_0')] = G_1'' + \alpha[\partial G_1''/\partial T + 2(F_0''G_0' - F_0G_0''')] + 2\beta(F_0''G_0' - F_0'G_0''), \quad \text{etc.} \quad (20)$$

3. Unbounded single plate

In this case the boundary conditions (7) are to be used which can be written as

$$F_m = F_m' = 0 \quad \text{at} \quad \eta = 0; \qquad F_m' \to 0 \quad \text{as} \quad \eta \to \infty \quad (\text{for } m = 0, 1, 2, \ldots), \quad (21)$$

$$G_0 = \cos T, \quad G_{m+1} = 0 \quad \text{at} \quad \eta = 0; \qquad G_m \to 0 \quad \text{as} \quad \eta \to \infty \quad (\text{for } m = 0, 1, 2, \ldots); \quad (22)$$

λ being zero, we get
$$\lambda_0 = \lambda_1 = \lambda_2 = 0,$$
and then the solution of the equations (17)–(20) satisfying the boundary conditions (21) and (22) is (d being $(2\nu_1/n)^{\frac{1}{2}}$, $R = 2$)
$$F_0(\eta, T) = 0, \tag{23}$$
$$G_0(\eta, T) = e^{-A\eta}\cos(T - B\eta), \tag{24}$$
$$F_1(\eta, T) = -\frac{1}{8A^3}\left(1 - \frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)(1 - 2A\eta - e^{-2A\eta})$$
$$+\frac{1}{8\sqrt{2}}\left\{\frac{1+9(\alpha+\beta)^2}{1+9a^2}\right\}^{\frac{1}{2}}[(1+\alpha^2)^{\frac{1}{4}}\{\cos(2T+\chi-\theta) - e^{-2A\eta}\cos(2T - 2B\eta + \chi - \theta)\}$$
$$-\sqrt{2}(1+4\alpha^2)^{\frac{1}{4}}\{\cos(2T+\chi-\phi) - e^{-C\eta}\cos(2T - D\eta + \chi - \phi)\}], \tag{25}$$
$$G_1(\eta, T) = 0, \tag{26}$$
where
$$A = \sqrt{2}\cos\theta/(1+\alpha^2)^{\frac{1}{4}}, \quad B = \sqrt{2}\sin\theta/(1+\alpha^2)^{\frac{1}{4}},$$
$$C = 2\cos\phi/(1+4\alpha^2)^{\frac{1}{4}}, \quad D = 2\sin\phi/(1+4\alpha^2)^{\frac{1}{4}},$$
$$\phi = \tan^{-1}[\{(1+4\alpha^2)^{\frac{1}{2}} - 2\alpha\}/\{(1+4\alpha^2)^{\frac{1}{2}} + 2\alpha\}]^{\frac{1}{2}},$$
$$\theta = \tan^{-1}[\{(1+\alpha^2)^{\frac{1}{2}} - \alpha\}/\{(1+\alpha^2)^{\frac{1}{2}} + \alpha\}]^{\frac{1}{2}},$$
$$\chi = \tan^{-1}[(1 - 9\alpha^2 - 9\alpha\beta)/(6\alpha + 3\beta)].$$
Having $F_0(\eta, T) = 0$, the first-order solution is a transverse velocity
$$v = r\Omega\exp\{-A(n/2\nu_1)^{\frac{1}{2}}z\}\cos[nt - B(n/2\nu_1)^{\frac{1}{2}}z]. \tag{27}$$
From the expression for v, the boundary-layer thickness is of the order of $(\nu_1/nA^2)^{\frac{1}{2}}$. Since μ_2 is negative and is small compared with μ_1 for the fluids so far studied, A decreases as n increases. If we study the flow of different second-order fluids for a particular value of n, we find that the boundary-layer thickness increases with the increase of $|\mu_2/\mu_1|$. If we choose a particular fluid and study the flow for different values of n, the boundary-layer thickness decreases as n increases provided $n < |\mu_1/3\mu_2|$. The condition of slow flow will be violated if n exceeds this value, hence for second-order fluids the boundary-layer thickness decreases with increase of n.

The radial and axial components of the velocity can be divided into steady and fluctuating parts. Let u_s, w_s denote the steady parts and u_f, w_f denote the fluctuating parts of u and w, respectively. We can write
$$u_s = \frac{r\Omega\epsilon}{4A^2}\left(1 - \frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)[1 - \exp\{-A(2n/\nu_1)^{\frac{1}{2}}z\}], \tag{28}$$
$$w_s = \frac{\Omega\epsilon}{4A^3}\left(\frac{2\nu_1}{n}\right)^{\frac{1}{2}}\left(1 - \frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)\left(1 - A\left(\frac{2n}{\nu_1}\right)^{\frac{1}{2}}z - \exp\{-A(2n/\nu_1)^{\frac{1}{2}}z\}\right), \tag{29}$$
$$u_f = \frac{r\Omega\epsilon}{4}\left[\frac{1+9(\alpha+\beta)^2}{1+9\alpha^2}\right]^{\frac{1}{2}}[\exp\{-A(2n/\nu_1)^{\frac{1}{2}}z\}\cos(2T - B(2n/\nu_1)^{\frac{1}{2}}z + \chi)$$
$$+\exp\{-C(n/2\nu_1)^{\frac{1}{2}}z\}\cos(2T - D(n/2\nu_1)^{\frac{1}{2}}z + \chi)], \tag{30}$$
$$w_f = \frac{\Omega\epsilon}{4}\left[\frac{\nu_1 + 9\nu_1(\alpha+\beta)^2}{2n + 18n\alpha^2}\right]^{\frac{1}{2}}[2(1+4\alpha^2)^{\frac{1}{4}}\{\cos(2T+\chi-\phi)$$
$$-\exp\{-C(n/2\nu_1)^{\frac{1}{2}}z\}\cos(2T + D(n/2\nu_1)^{\frac{1}{2}}z + \chi - \phi)\} - \sqrt{2}(1+\alpha^2)^{\frac{1}{4}}$$
$$\times\{\cos(2T+\chi-\theta) - \exp\{-A(2n/\nu_1)^{\frac{1}{2}}z\}\cos(2T - B(2n/\nu_1)^{\frac{1}{2}}z + \chi - \theta)\}]. \tag{31}$$

From (30) and (31) we deduce that u_f decreases as z increases and vanishes outside the boundary layer, while

$$w_f(\infty) = \frac{\Omega\epsilon}{4}\left[\frac{\nu_1+9\nu_1(\alpha+\beta)^2}{2n+18n\alpha^2}\right]^{\frac{1}{2}}$$
$$\times[2(1+4\alpha^2)^{\frac{1}{4}}\cos(2T+\chi-\phi)-\surd 2(1+\alpha^2)^{\frac{1}{4}}\cos(2T+\chi-\theta)] \quad (32)$$

which is necessary to maintain the continuity of the flow. The form of u_s and w_s suggests the definition of a stream function as

$$\psi_s = \frac{r^2\Omega\epsilon}{4A^3}\left(\frac{\nu_1}{2n}\right)^{\frac{1}{2}}\left[A\left(\frac{2n}{\nu_1}\right)^{\frac{1}{2}}z-1+\exp\{-A(2n/\nu_1)^{\frac{1}{2}}z\}\right]\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right),$$

which at large distances from the plate behaves as

$$\psi_s(\infty) = \frac{r^2\Omega\epsilon}{4A^2}\left[z-\left(\frac{\nu_1}{2A^2n}\right)^{\frac{1}{2}}\right]\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right). \quad (33)$$

This is the stream function of an axially symmetric flow against an imaginary wall $z = (\nu_1/2A^2n)^{\frac{1}{2}}$. The persistence of u_s outside the boundary-layer region indicates an error which is due to approximating the inertia force by its centrifugal part even at large distances from the plate in this type of series expansion. A further investigation of the steady part of the flow is required.

4. Presence of another plate

In this case the boundary conditions (8) are to be satisfied, these can be written as

$$F_m = F'_m = 0 \text{ at } \eta = 0; \quad F_m = F'_m = 0 \text{ at } \eta = 1 \quad (\text{for } m = 0,1,2,\ldots); \quad (34)$$
$$G_0 = \cos T,\ G_{m+1} = 0 \text{ at } \eta = 0; \quad G_m = 0 \text{ at } \eta = 1 \quad (\text{for } m = 0,1,2,\ldots). \quad (35)$$

The solution of the equations (17) and (19), satisfying the above boundary conditions, is

$$F_0(\eta, T) = 0, \quad \lambda_0(T) = 0, \quad (36)$$

$$G_0(\eta,T) = \frac{\begin{array}{c}[\cosh\{A(2-\eta)\}\cos(B\eta)-\cosh(A\eta)\cos\{B(2-\eta)\}]\cos(nt)\\ +[\sinh\{A(2-\eta)\}\sin(B\eta)-\sinh(A\eta)\sin\{B(2-\eta)\}]\sin(nt)\end{array}}{[\cosh(2A)-\cos(2B)]}, \quad (37)$$

where

$$A = [R\{(1+\alpha^2)^{\frac{1}{2}}+\alpha\}/2(1+\alpha^2)]^{\frac{1}{2}}, \quad B = [R\{(1+\alpha^2)^{\frac{1}{2}}-\alpha\}/2(1+\alpha^2)]^{\frac{1}{2}}.$$

For small values of R, the transverse component of the velocity behaves as

$$\frac{v}{r\Omega} = (1-\eta)\left[\left\{1-\frac{R\alpha}{6(1+\alpha^2)}\eta(2-\eta)-\frac{R^2(1-\alpha^2)}{360(1+\alpha^2)^2}\eta(8+8\eta-12\eta^2+3\eta^3)\right\}\cos(nt)\right.$$
$$\left.+\frac{R\eta(2-\eta)}{6(1+\alpha^2)}\left\{1+\frac{R\alpha}{30(1+\alpha^2)}(-4-6\eta+3\eta^2)\right\}\sin(nt)\right]+O(R^3). \quad (38)$$

For large values of R it behaves as

$$v/r\Omega = e^{-A\eta}\cos(nt-B\eta), \quad (39)$$

which is similar to the expression for $v/r\Omega$ in the case of the unbounded plate.

The function G_1 is zero throughout and the function F_1 and the constant λ_1 are of the following form,

$$F_1(\eta, T) = f(\eta) + h(\eta)\, e^{2iT}, \quad \lambda_1(T) = K_1 + M_1 e^{2iT}.$$

This form is suggested by G_0^2 which is composed of a steady term and a term of frequency $2n$. Complex notation has been adopted here with the convention that only the real parts of the complex quantities have physical meaning. Substituting the expressions for F_0, F_1, G_0, λ_1 into (18) and equating the coefficients of e^{2iT} and terms independent of it, we get two equations giving $f(\eta)$ and $h(\eta)$. The function $f(\eta)$ and the constant K_1 are given by

$$\begin{aligned} f(\eta) = \frac{RA^{-3}B^{-3}(1+\alpha^2)^{-\frac{1}{2}}}{16(\cosh 2A - \cos 2B)} & [\{4\alpha + 3\beta - (1+\alpha^2)^{\frac{1}{2}}\}\{(1-3\eta^2+2\eta^3)\sinh 2A \\ & - \sinh(2A[1-\eta]) + 2A\eta^2(1-\eta) - 2A\eta(1-\eta)^2 \cosh 2A\} B^3 \\ & - \{4\alpha + 3\beta + (1+\alpha^2)^{\frac{1}{2}}\}\{(1-3\eta^2+2\eta^3)\sin 2B - \sin(2B[1-\eta]) \\ & + 2B\eta^2(1-\eta) - 2B\eta(1-\eta)^2\cos 2B\} A^3], \end{aligned} \tag{40}$$

$$\begin{aligned} K_1 = -\frac{3RA^{-3}B^{-3}(1+\alpha^2)^{-\frac{1}{2}}}{8(\cosh 2A - \cos 2B)} & [\{4\alpha + 3\beta - (1+\alpha^2)^{\frac{1}{2}}\}\{A\cosh(2A) - \sinh(2A) + A\} B^3 \\ & - \{4\alpha + 3\beta + (1+\alpha^2)^{\frac{1}{2}}\}(B\cos 2B - \sin 2B + B) A^3]. \end{aligned} \tag{41}$$

For small values of R, the steady parts of u and w to the second order of approximation in ϵ are given by

$$\begin{aligned} \frac{u_s}{r\Omega} = & \frac{\epsilon R(1-3\alpha^2-3\alpha\beta)}{120(1+\alpha^2)} \eta(1-\eta)(6-15\eta+5\eta^2) \\ & - \frac{\epsilon R^2 \alpha\eta(1-\eta)}{2520(1+\alpha^2)} \left[2 + 35\eta - 85\eta^2 + 35\eta^3 - 7\eta^4 + \frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}} \left\{ (20 - 70\eta + 70\eta^2 \right.\right. \\ & \left.\left. + 35\eta^3 + 7\eta^4)\left(\frac{2A^4+2B^4}{A^4-B^4}\right) - (6-15\eta+5\eta^2)\left(\frac{7A^2-7B^2}{A^2+B^2}\right)\right\}\right] + O(R^3) \end{aligned} \tag{42}$$

and

$$\begin{aligned} \frac{w_s}{d\Omega} = & \frac{\epsilon R(1-3\alpha^2-3\alpha\beta)}{60(1+\alpha^2)} \eta^2(1-\eta)^2(3-\eta) \\ & - \frac{\epsilon R^2\alpha\eta^2(1-\eta)^2}{1260(1+\alpha^2)} \left[1 + 13\eta - 5\eta^2 + \eta^3 + \frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}} \right. \\ & \left. \times \left\{ (10 - 10\eta + 5\eta^2 - \eta^3)\left(\frac{2A^4+2B^4}{A^4-B^4}\right) - (3-\eta)\left(\frac{7A^2-7B^2}{A^2+B^2}\right)\right\}\right] + O(R^3), \end{aligned} \tag{43}$$

respectively. The asymtotic behaviour of u_s, ω_s for large values of R is given by

$$\frac{u_s}{r\Omega} \sim \frac{\epsilon}{4A}\left\{\frac{1+\alpha^2-(4\alpha+3\beta)(1+\alpha^2)^{\frac{1}{2}}}{((1+\alpha^2)^{\frac{1}{2}}+\alpha)}\right\}[(1-\eta)(1-3\eta)A + 3\eta(1-\eta) - A\,e^{-2A\eta}], \tag{44}$$

$$\frac{w_s}{d\Omega} \sim \frac{\epsilon}{4A}\left\{\frac{1+\alpha^2-(4\alpha+3\beta)(1+\alpha^2)^{\frac{1}{2}}}{((1+\alpha^2)^{\frac{1}{2}}+\alpha)}\right\}[(1-\eta)^2(1+2\eta) - 2A\eta(1-\eta)^2 - e^{-2A\eta}]. \tag{45}$$

The expression for $h(\eta)$ is complicated and it is cumbersome to separate its real and imaginary parts. However, the asymtotic behaviour of $h(\eta)$ and $h'(\eta)$ for large values of R can be written as

$$\frac{4[2\alpha-i(1+3\alpha^2)]}{1-3\alpha(\alpha+\beta)-i(4\alpha+3\beta)}h(\eta)=\frac{2(A+iB)-(C+iD)}{2(C+iD-2)(A+iB)}$$
$$\times\left[\eta-1+\frac{1+e^{-(C+iD)\eta}-e^{-(C+iD)(1-\eta)}}{C+iD}\right]+\frac{e^{-(C+iD)\eta}}{C+iD}-\frac{e^{-2(A+iB)\eta}}{2(A+iB)} \tag{46}$$

and
$$\frac{4[2\alpha-i(1+3\alpha^2)]}{1-3\alpha(\alpha+\beta)-i(4\alpha+3\beta)}h'(\eta)=\frac{2(A+iB)-(C+iD)}{2(C+iD-2)(A+iB)}$$
$$\times[1-e^{-(C+iD)(1-\eta)}-e^{-(C+iD)\eta}]+e^{-2(A+iB)\eta}-e^{-(C+iD)\eta}, \tag{47}$$

where

$$C=[R\{(1+4\alpha^2)^{\frac{1}{2}}+2\alpha\}/(1+4\alpha^2)]^{\frac{1}{2}},\quad D=[R\{(1+4\alpha^2)^{\frac{1}{2}}-2\alpha\}/(1+4\alpha^2)]^{\frac{1}{2}}.$$

The asymtotic behaviour of the constants is given by

$$K_1\sim 3A^{-1}(A-1)[(1+\alpha^2)-(4\alpha+3\beta)(1+\alpha^2)^{\frac{1}{2}}]/[4(1+\alpha^2)^{\frac{1}{2}}+\alpha],$$
$$M_1\sim[2(A+iB)-(C+iD)][1-3\alpha^2-3\alpha\beta-i(4\alpha+3\beta)]/[2\alpha-i(1+3\alpha^2)]$$
$$\times[8R(A+iB)(C+iD-2)].$$

The expressions for $h(\eta)$ and $h'(\eta)$ contain the terms of the type $e^{-l\eta}$ as well as $e^{-k(1-\eta)}$ where l and k are constants. This fact shows that the boundary layer starts developing at both the plates as R increases.

5. Discussion

In the case of oscillation of an unbounded plate (fluid from $z=0$ to $z=\infty$) the transverse shearing stress at the plate is given by

$$[\tau_{z\theta}]_{z=0}=-r\rho\Omega(\nu_1 n)^{\frac{1}{2}}(1+\alpha^2)^{\frac{1}{4}}\cos[T+\tfrac{1}{2}\pi-\theta].$$

$G_1(\eta,T)$ being zero, it is correct up to second order in ϵ and the correction of the third order in ϵ can be shown to be negligibly small (Rosenblat 1959). It has a phase lead of $[\frac{1}{2}\pi-\theta]$ over the oscillation of the plate and this lead decreases as n increases. For a 6·8% solution of poly-iso-butylene in cetane at 30 °C the phase leads are 44·23°, 43·85°, 43·47° for n = 1·0, 1·5, 2·0 respectively. For Newtonian viscous fluid, i.e. when μ_2 and μ_3 are taken to be zero, this phase lead is independent of n and is of 45·00°. The radial shearing stress is given by

$$[\tau_{zr}]_{z=0}=\frac{r\Omega\epsilon(2\nu_1 n)^{\frac{1}{2}}}{4}\left[\frac{1}{A}\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)\right.$$
$$+\left\{\frac{1+9(\alpha+\beta)^2}{1+9\alpha^2}\right\}^{\frac{1}{2}}\left\{(1+4\alpha^2)^{\frac{1}{4}}\cos(2T+\chi+\tfrac{1}{2}\pi-\phi)\right.$$
$$\left.\left.-\frac{\sqrt{2}(1+4\alpha^2)^{\frac{1}{4}}}{(1+\alpha^2)^{\frac{1}{4}}}\cos(2T+\chi+\tfrac{1}{2}\pi+\theta-2\phi)\right\}\right].$$

This shearing stress is composed of a steady and a fluctuating part and is of second order in ϵ.

In the case when a stationary plate is present at $z = d$ the transverse shearing stresses on the plates for small values of R are given by

$$[\tau_{z\theta}]_{z=0} = -\frac{r\Omega\mu_1}{d}\left[\left\{1+\frac{R^2}{45(1+\alpha^2)}\right\}\cos(nt) - \left\{\alpha+\frac{R}{3}-\frac{R^2\alpha}{45(1+\alpha^2)}\right\}\sin(nt)\right],$$

$$[\tau_{z\theta}]_{z=d} = -\frac{r\Omega\mu_1}{d}\left[\left\{1-\frac{7R^2}{360(1+\alpha^2)}\right\}\cos(nt) - \left\{\alpha-\frac{R}{6}+\frac{7R^2\alpha}{360(1+\alpha^2)}\right\}\sin(nt)\right].$$

It is obvious from the above expressions that the shearing stress on the stationary plate tends to that on the oscillating one when R tends to zero. For large values of R the shearing stresses on the plates behave as

$$[\tau_{z\theta}]_{z=0} = -\frac{r\Omega\mu_1}{d}[(A-\alpha B)\cos(nt) - (B+\alpha A)\sin(nt)],$$

$$[\tau_{z\theta}]_{z=d} = -\frac{r\Omega\mu_1}{d}[A+B+\alpha(A-B)]\,e^{-A} \simeq 0.$$

When R tends to infinity the shearing stress on the stationary plate tends to zero and the shearing stress on the oscillating plate behaves as if the stationary plate were absent.

At this stage a further investigation of the steady part of the flow in the case when the fluid is infinite in extent (single plate) is considered by another approximate Pohlhausen-type method in which all the inertia terms are retained. Replacing F by F_s and G, G' by their root-mean squares

$$\bar{G} = 2^{-\frac{1}{2}}\,e^{-A\eta}, \quad \bar{G}' = (1+\alpha^2)^{-\frac{1}{4}}\,e^{-A\eta}$$

respectively in equation (16) we get

$$\epsilon^2(F_s'^2 - 2F_sF_s'') = \tfrac{1}{2}F_s''' - \alpha\epsilon^2(F_s''^2 + F_sF_s^{\text{iv}}) - \tfrac{1}{2}\beta\epsilon^2(F_s''^2 + 2F_s'F''') + \tfrac{1}{2}\{1-(4\alpha+3\beta)(1+\alpha^2)^{-\frac{1}{2}}\}\,e^{-2A\eta}. \qquad (48)$$

Taking δ to be the dimensionless boundary-layer thickness and integrating (48) from 0 to δ we have

$$6\epsilon^2\int_0^\delta F_s'^2\,d\eta = -F_s''(0) + \epsilon^2(\beta-4\alpha)\int_0^\delta F_s''^2\,d\eta + \frac{1}{2A}\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right) \qquad (49)$$

with boundary conditions

$$\left.\begin{array}{l} F_s(0) = 0,\; F_s'(0) = 0,\; F_s'''(0) + \{1-(4\alpha+3\beta)(1+\alpha^2)^{-\frac{1}{2}}\} = \epsilon^2(2\alpha+\beta)\,F_s''(0), \\ F_s'(\delta) = F_s''(\delta) = F_s'''(\delta) = \ldots = 0. \end{array}\right\} \qquad (50)$$

A solution of (49) may be taken of the form

$$F_s' = -\frac{1}{4A^2}\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)(e^{-2A\eta} - e^{-\eta/\delta}). \qquad (51)$$

If we neglect δ^{-2} and terms of similar order the boundary conditions are satisfied and when $\eta \ll 1$, $e^{-\eta/\delta} \doteqdot 1$; hence near the plate F_s' in (51) behaves as $u_s/r\Omega\epsilon$ in (28). Substituting (51) into (49), we get

$$\{1-(4\alpha+3\beta)(1+\alpha^2)^{-\frac{1}{2}}\}(2A\delta-1)^2(3A\delta+4\alpha A^2-\beta A^2) = (8A^3/\epsilon^2)(2A\delta+1), \qquad (52)$$

which gives δ. Integrating (51) and applying the boundary conditions (50) we have

$$F_s = \frac{1}{8A^3}\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)[2A\delta(1-e^{-\eta/\delta})-(1-e^{-2A\eta})], \tag{53}$$

$$F_s(\infty) = \frac{1}{8A^3}\left(1-\frac{4\alpha+3\beta}{(1+\alpha^2)^{\frac{1}{2}}}\right)(2A\delta-1). \tag{54}$$

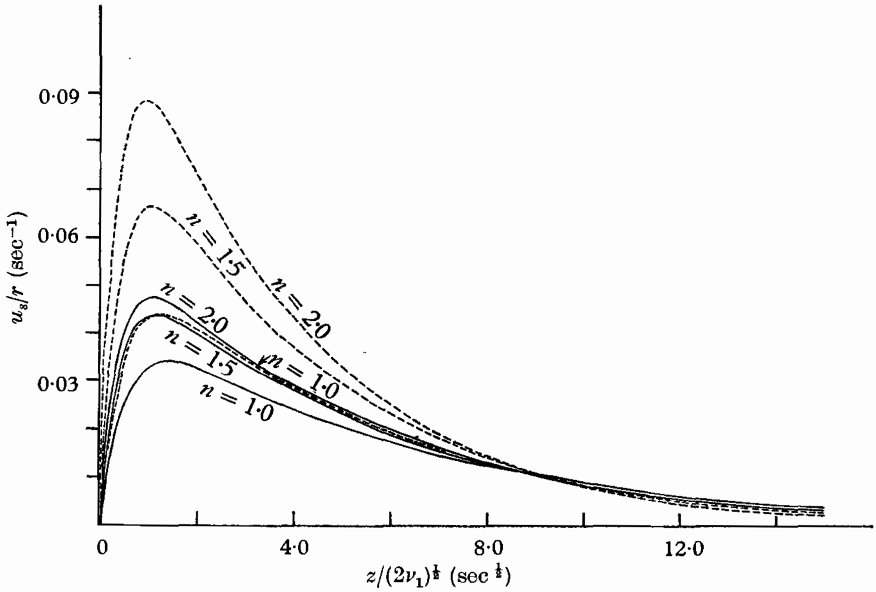

FIGURE 1. Steady part of radial velocity, plotted as u_s/r as a function of distance from the plate plotted as $z/(2\nu_1)^{\frac{1}{2}}$, with $\epsilon = \frac{1}{4}$. ——, Second-order fluid; - - -, corresponding case in Newtonian fluid.

For a 6·8% solution of poly-iso-butylene in cetane at 30 °C, $\mu_1 = 60{\cdot}0$, $\mu_2 = -1{\cdot}6$, $\mu_3 = 7{\cdot}4$ (all expressed in CGS units) and taking $\epsilon = \frac{1}{4}$ we obtain the results shown in table 1. The non-dimensional boundary-layer thickness δ decreases as n in-

n	α	β	δ	$F_s(\infty)$
1·0	−0·0267	0·1233	6·0371	1·0468
1·5	−0·0400	0·1850	6·6194	0·9637
2·0	−0·0533	0·2467	7·3307	0·9190
Newtonian fluids	zero	zero	5·3306	1·2077

TABLE 1.

creases while in the case of Newtonian fluids it is independent of n. Consequently the rate of decrease of boundary-layer thickness $(2\nu_1\delta^2/n)^{\frac{1}{2}}$ is much less than that in Newtonian fluids. The axial component of the velocity at infinity $w_s(\infty)$ increases with n but the rate of increase is much less than that in Newtonian fluids; again this fact is due to decrease of $F_s(\infty)$ as n increases. The graphs for

u_s/r and $-w_s/\nu_1^{\frac{1}{2}}$ have been plotted against $z/(2\nu_1)^{\frac{1}{2}}$ for $n = 1{\cdot}0, 1{\cdot}5, 2{\cdot}0$ in figures 1 and 2 respectively. The lines representing the corresponding flow in Newtonian fluids, i.e. taking $\mu_2 = \mu_3 = 0$ are shown in the graph by the dotted lines.

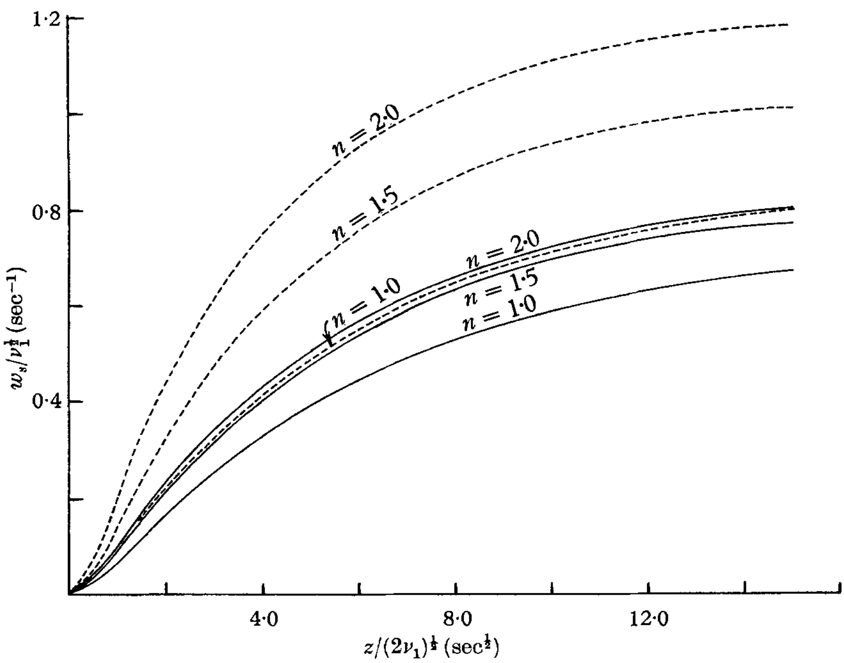

FIGURE 2. Steady part of axial velocity, plotted as $w_s/\nu_1^{\frac{1}{2}}$ as a function of distance from the plate plotted as $z/(2\nu_1)^{\frac{1}{2}}$, with $\epsilon = \frac{1}{4}$. ——, Second-order fluid; – – –, corresponding case in Newtonian fluid.

I thank Prof. H. Markovitz for supplying the data, Prof. S. L. Soo for suggestions and U.S.A.I.D. for financial assistance during this work.

REFERENCES

COLEMAN, B. D. & NOLL, W. 1960 *Arch. Rat. Mech. Anal.* **6**, 355.
NOLL, W. 1958 *Arch. Rat. Mech. Anal.* **2**, 197.
RIVLIN, R. S. & ERICKSEN, J. L. 1955 *J. Rat. Mech. Anal.* **4**, 323.
ROSENBLAT, S. 1959 *J. Fluid Mech.* **6**, 206.
ROSENBLAT, S. 1960 *J. Fluid Mech.* **8**, 388.
SRIVASTAVA, A. C. 1959 *Appl. Sci. Res.* A, **9**, 369.
SRIVASTAVA, A. C. 1960 *Proc. of the Symposium on Fluid Dynamics, Indian Science Congress* (Bombay, India).